

# Optimal entanglement manipulation via coherent-state transmission

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We derive an optimal bound for arbitrary entanglement manipulation based on the transmission of a pulse in coherent states over a lossy channel followed by local operations and unlimited classical communication (LOCC). This stands on a theorem to reduce LOCC via a local unital qubit channel to local filtering. We also present an optimal protocol based on beam splitters and a quantum nondemolition (QND) measurement on photons. Even if we replace the QND measurement with photon detectors, the protocol outperforms known entanglement generation schemes.

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Entanglement is now well known as an essential resource for quantum communication [1] despite it being found in an attempt to point out a paradoxical nature of quantum mechanics [2]. In fact, it is known that any quantum communication [including quantum key distribution (in the sense of Ref. [3])] can never be accomplished by distant parties who are not capable of sharing entangled pairs. This implies the importance of evaluating the potential to share entanglement through a given communication channel, which determines its value as a quantum channel. If we look at practical quantum communication such as fiber-based quantum key distribution, free-space quantum communication, entanglement generation in quantum repeaters, quantum communication via superconducting transmission lines, and a quantum memory for bosons (transmission in time), we become aware that all the protocols rely on a lossy bosonic channel. Thus, quantum communication based on this channel is practically the most important class (cf. [4]).

One of the most fundamental protocols in this class is the family of coherent-state-based protocols represented by Bennett 1992 quantum key distribution [5] and entanglement generation protocols in quantum repeaters [6–11]. These protocols are based on the transmission of a pulse in coherent states over a lossy channel, and they are dominated by the following paradigm: (i) A sender prepares an entangled state composed of computational basis states of a qubit  $A$  and coherent states of a pulse  $a$ . (ii) The sender then sends the pulse  $a$  to the mode  $b$  at the receiver's site through a lossy channel. (iii) Then, the sender and the receiver manipulate the shared system  $Ab$  through their local operations and unlimited two-way classical communication (LOCC) in order to convert the initial entangled state to a more entangled state by tolerating failure. Hence, the potential of the coherent-state-based protocols is determined by optimizing the LOCC manipulation for a single entangled pair  $Ab$ . This kind of “entanglement manipulation” is completely understood for a pure-state input  $Ab$  [12]. But, the analysis for a mixed-state input  $Ab$  as considered here has remained a long-standing open question

[12] despite its significance. In addition, the LOCC manipulation is beyond the paradigms in Refs. [8, 10, 13]. Therefore, grasping the potential of such coherent-state-based protocols must be a key step in the practical and theoretical evolution of quantum communication.

In this paper, we present a theoretical limit of the performance of arbitrary coherent-state-based protocols, as well as a simple protocol that achieves the limit. This is based on a general proposition to reduce LOCC manipulation via a local unital qubit channel to local filtering. The derived limit is represented in terms of the total success probability and an average entanglement monotone [14] of the generated entangled states, and it is determined only by the transmittance of the channel. The bound is shown to be accomplished by a simple protocol composed only of beam splitters and a quantum nondemolition (QND) measurement [15] on photons. If we substitute photon-number-resolving detectors for the QND measurement, the protocol can entangle distant qubits with near-optimal performance, which is shown to outperform known protocols [6–10]. Hence, these protocols play the role of an efficient entanglement supplier for various quantum communication schemes.

*Coherent-state-based protocols.*—We start by defining the protocols considered here: (A-i) A sender called Alice prepares a qubit  $A$  and a pulse  $a$  in her desired state in the form of  $\sum_{j=0,1} e^{i\Theta_j} \sqrt{q_j} |j\rangle_A |\alpha_j\rangle_a$  for a computational basis  $\{|j\rangle_A\}_{j=0,1}$ , coherent states  $\{|\alpha_j\rangle_a\}_{j=0,1}$ , real parameters  $\Theta_j$ , and  $q_j \geq 0$  with  $\sum_{j=0,1} q_j = 1$ ; (A-ii) Alice sends the pulse  $a$  to a receiver called Bob, through a lossy channel described by an isometry  $|\alpha\rangle_a \rightarrow |\sqrt{T}\alpha\rangle_b |\sqrt{1-T}\alpha\rangle_e$ , where  $T$  is the transmittance,  $b$  is a mode at Bob's place, and  $e$  is the environment; (A-iii) Then, Alice and Bob manipulate the system  $Ab$  through LOCC to obtain an entangled state  $\hat{\tau}_k^{A'B}$  between Alice's system  $A'$  and Bob's system  $B$ , and declare whether they obtain a success outcome  $k$  occurring with a probability  $p_k$  or a failure outcome. Note that the output systems  $A'B$  are not limited to qubits [16]. In what follows, the set of all the success events  $k$  is denoted by  $\mathcal{S}$ .

As a measure of the performance of the protocols, we

take the total success probability, i.e.,  $P_s = \sum_{k \in \mathcal{S}} p_k$ . We also need to choose an entanglement measure for estimating the value of the obtained entangled states  $\{\hat{\tau}_k^{A'B}\}_{k \in \mathcal{S}}$ . Since the output system  $A'B$  has no restrictions in contrast to those described in Refs. [8, 10, 13], the singlet fraction may be unsuitable. Thus, here we take an entanglement monotone  $E$  [14] that is a convex monotonically nondecreasing function of the concurrence  $C$  [17] at least for qubits (cf. [18]). Based on this  $E$ , as another measure of the protocols, we adopt the average  $\bar{E}$  of the obtained entangled states  $\{\hat{\tau}_k^{A'B}\}_{k \in \mathcal{S}}$ , namely  $\bar{E} = [\sum_{k \in \mathcal{S}} p_k E(\hat{\tau}_k^{A'B})]/P_s$ .

We also allow Alice and Bob to switch among two or more protocols probabilistically. This corresponds [13] to taking the convex hull of achievable points  $(P_s, P_s \bar{E})$ .

*Virtual protocol.*—For an actual protocol, we define the virtual protocol [10] that works in the same way as the actual protocol but simplifies the analysis significantly. Steps (A-i) and (A-ii) indicate that, when the pulse arrives at Bob's site, the state of the total system  $Abe$  is written in the form  $|\psi\rangle_{Abe} = \sum_{j=0,1} \sqrt{q_j} |j\rangle_A |u_j\rangle_b |v_j\rangle_e$  for states  $\{|u_j\rangle\}_{j=0,1}$  and  $\{|v_j\rangle\}_{j=0,1}$  with  $|\langle u_1 | u_0 \rangle|^{1-T} = |\langle v_1 | v_0 \rangle|^T > 0$ . Thus, for a state  $|\psi'\rangle_{Ab} := \sum_{j=0,1} \sqrt{q_j} e^{i(-1)^j \xi} |j\rangle_A |u_j\rangle_b$  with  $2\xi := \arg[\langle v_1 | v_0 \rangle]$  and for a phase-flip channel  $\Lambda_u^A(\rho) := f_u \hat{\rho} + (1 - f_u) \hat{Z}^A \hat{\rho} \hat{Z}^A$  with  $\hat{Z}^A := |0\rangle\langle 0|_A - |1\rangle\langle 1|_A$  and  $f_u := (1 + u^{\frac{1-T}{T}})/2$ , we have  $\text{Tr}_e[|\psi\rangle\langle\psi|_{Abe}] = \Lambda_{\langle u_1 | u_0 \rangle}^A(|\psi'\rangle\langle\psi'|_{Ab})$ . Hence, we can consider any protocol to have the following sequence: (V-i) System  $Ab$  is prepared in  $|\psi'\rangle_{Ab}$ ; (V-ii)  $\Lambda_{\langle u_1 | u_0 \rangle}^A$  is applied on qubit  $A$ ; (V-iii) Alice and Bob perform an LOCC, which provides  $\hat{\tau}_k^{A'B}$ . We call this sequence “the virtual protocol.”

We introduce a proposition that enables us to derive an optimal bound in more general settings (cf. [22]).

*Proposition.*—Let  $(P_s, \bar{E})$  be the performance of an LOCC protocol starting with qubits  $AB$  in state  $\mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB})$ , where  $\mathcal{E}^A$  is a random local unitary channel [20] defined by  $\mathcal{E}^A(\hat{\rho}^{AB}) := \sum_l q_l \hat{U}_l^A \hat{\rho}^{AB} (\hat{U}_l^A)^\dagger$ . Then, there is a protocol that is not less efficient than  $(P_s, \bar{E})$  but that is based only on Bob's measurement. In addition, for Schmidt coefficients  $\lambda_0$  and  $\lambda_1 (\leq \lambda_0)$  of  $|\varphi\rangle_{AB}$ , the achievable region of  $(P_s, P_s \bar{E})$  is described by the convex hull of  $\{(P_s, P_s \bar{E}) \mid 0 \leq P_s \leq 1, 0 \leq \bar{E} \leq E(C^{\max}(P_s))\}$  with  $C^{\max}(P_s) := (2\sqrt{\lambda_0 \lambda_1})^{-1} C(\mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB}))$  for  $P_s < 2\lambda_1$  and  $C^{\max}(P_s) := P_s^{-1} \sqrt{(P_s - \lambda_1)/\lambda_0} C(\mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB}))$  for  $P_s \geq 2\lambda_1$ .

*Proof.* Let Kraus operators  $\{\hat{M}_k^A \otimes \hat{N}_k^B\}_{k \in \mathcal{S}}$  be Alice and Bob's successful measurement in step (V-iii). Without loss of generality, the input spaces of  $\hat{M}_k^A$  and  $\hat{N}_k^B$  can be assumed to be qubit spaces. If Alice and Bob can achieve the measurement  $\{\hat{M}_k^A \otimes \hat{N}_k^B\}_{k \in \mathcal{S}}$ , they can always, in principle, obtain a state  $\hat{\tau}_k^{AB} := (\hat{M}_k^A \otimes \hat{N}_k^B) \mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB}) (\hat{M}_k^A \otimes \hat{N}_k^B)^\dagger / p_k$ . From the con-

vexity of the entanglement monotone  $E$  [14], the performance of this protocol is not less than protocols where, for a set  $\mathcal{S}' \subset \mathcal{S}$ , they provide a mixture of the states  $(\sum_{k \in \mathcal{S}'} p_k \hat{\tau}_k^{AB}) / (\sum_{k \in \mathcal{S}'} p_k)$  instead of states  $\{\hat{\tau}_k^{AB}\}_{k \in \mathcal{S}'}$ . Thus, we can assume that Alice and Bob return the state  $\hat{\tau}_k^{AB}$  with probability  $p_k$ . Note that the range of  $\hat{\tau}_k^{AB}$  can be assumed to be qubit spaces.

From Proposition 1 in Ref. [23], for any  $\hat{U}_l^A$ , there exist unitary operators  $\{\hat{V}_{k|l}^A\}_k$  and Kraus operators  $\{\hat{O}_{k|l}^B\}_k$  that satisfy  $(\hat{M}_k^A \hat{U}_l^A \otimes \hat{N}_k^B) |\varphi\rangle_{AB} = (\hat{V}_{k|l}^A \hat{U}_l^A \otimes \hat{O}_{k|l}^B) |\varphi\rangle_{AB}$  with  $d_k := \det[(\hat{M}_k^A)^\dagger \hat{M}_k^A] \det[(\hat{N}_k^B)^\dagger \hat{N}_k^B] = \det[(\hat{O}_{k|l}^B)^\dagger \hat{O}_{k|l}^B]$ . On the other hand, using the formula [17], we can show that the concurrence  $C$  for the state  $\hat{\tau}_k^{AB}$  is described by  $p_k C(\hat{\tau}_k^{AB}) = \sqrt{d_k} C(\mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB}))$ . Thus, if Bob performs  $\{\hat{O}_{k|l}^B\}_k$ , he obtains a state  $\hat{\tau}_{k|l}^{AB} := \hat{O}_{k|l}^B \mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB}) (\hat{O}_{k|l}^B)^\dagger / p_{k|l}$  with probability  $p_{k|l} := \langle \varphi | (\hat{O}_{k|l}^B)^\dagger \hat{O}_{k|l}^B | \varphi \rangle$  and concurrence  $C(\hat{\tau}_{k|l}^{AB}) = \sqrt{d_k} C(\mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB})) / p_{k|l} = p_k C(\hat{\tau}_k^{AB}) / p_{k|l}$ . Since  $\sum_l q_l p_{k|l} = p_k$  holds and  $\sum_l q_l p_{k|l} E(\hat{\tau}_{k|l}^{AB}) \geq p_k E(\hat{\tau}_k^{AB})$  is implied by  $\sum_l q_l p_{k|l} C(\hat{\tau}_{k|l}^{AB}) = p_k C(\hat{\tau}_k^{AB})$  and the convexity of  $E(C)$ , the original LOCC protocol is concluded to be outperformed by a protocol that performs only Bob's measurement  $\{\hat{O}_{k|l}^B\}_k$  with probability  $q_l$  and returns  $k$  and  $l$  as the outcome.

Thus, we focus on a protocol that is based on Bob's measurement  $\{\hat{O}_k^B\}_{k \in \mathcal{S}}$  and returns state  $\hat{\rho}_k^{AB} := \hat{O}_k^B \mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB}) (\hat{O}_k^B)^\dagger / p_k$  with probability  $p_k$ . We note that there are Kraus operators  $\hat{\Omega}^B$  and  $\{\hat{L}_k^B\}_{k \in \mathcal{S}}$  satisfying  $\hat{L}_k^B \hat{\Omega}^B = \hat{O}_k^B$ . In fact, if we define them as  $\hat{\Omega}^B := [\sum_{k \in \mathcal{S}} (\hat{O}_k^B)^\dagger \hat{O}_k^B]^{1/2}$  and  $\hat{L}_k^B := \hat{O}_k^B (\hat{\Omega}^B)^{-1}$ , where  $\hat{\Omega}^{-1}$  is the inverse of  $\hat{\Omega}$  in its range, the operators satisfy  $(\hat{\Omega}^B)^\dagger \hat{\Omega}^B \leq \hat{1}^B$  and  $\sum_{k \in \mathcal{S}} (\hat{L}_k^B)^\dagger \hat{L}_k^B \leq \hat{1}^B$  from  $\sum_{k \in \mathcal{S}} (\hat{O}_k^B)^\dagger \hat{O}_k^B \leq \hat{1}^B$ . Hence, we can regard Bob's measurement  $\{\hat{O}_k^B\}_{k \in \mathcal{S}}$  as a sequential measurement of  $\hat{\Omega}^B$  followed by  $\{\hat{L}_k^B\}_{k \in \mathcal{S}}$ . On the other hand, the entanglement monotone  $E$  of the state  $\tau_s^{AB} := \hat{\Omega}^B \mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB}) \hat{\Omega}^B / P_s$  with  $P_s = \sum_{k \in \mathcal{S}} p_k$  is not less than  $[\sum_{k \in \mathcal{S}} p_k E(\hat{\rho}_k^{AB})] / P_s$ , because the entanglement monotone  $E$  does not increase through a local operation on average [14]. Therefore, we can assume that Bob merely applies a filter  $\hat{\Omega}^B$  to qubits  $AB$ .

Let us proceed to the optimization of  $(P_s, E(\tau_s^{AB}))$  over the filter  $\hat{\Omega}^B$ . From the monotonicity of  $E(C)$ , our attention is concentrated on the maximization of  $C(\tau_s^{AB})$  for a fixed  $P_s$ . On the other hand, for the Schmidt decomposition of  $|\varphi\rangle_{AB} = \sum_{j=0,1} \sqrt{\lambda_j} |jj\rangle_{AB}$ , we have  $P_s = \langle \varphi | (\hat{\Omega}^B)^\dagger \hat{\Omega}^B | \varphi \rangle = \sum_{j=0,1} \lambda_j \langle j | (\hat{\Omega}^B)^\dagger \hat{\Omega}^B | j \rangle$  and  $P_s C(\tau_s^{AB}) = (\det[(\hat{\Omega}^B)^\dagger \hat{\Omega}^B])^{1/2} C(\mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB})) \leq (\prod_{j=0,1} \langle j | (\hat{\Omega}^B)^\dagger \hat{\Omega}^B | j \rangle)^{1/2} C(\mathcal{E}^A(|\varphi\rangle\langle\varphi|_{AB}))$ , where the equalities hold by choosing  $\hat{\Omega}^B$  with  $\langle 0 | (\hat{\Omega}^B)^\dagger \hat{\Omega}^B | 1 \rangle = 0$ . Combined with  $(\hat{\Omega}^B)^\dagger \hat{\Omega}^B \leq \hat{1}^B$ , this shows that  $C^{\max}$  is the maximum of  $C(\tau_s^{AB})$ . By considering the mixture of

protocols, the overall statement becomes the proposition.

*Optimal bound.*—Let us apply the proposition to our problem. Schmidt coefficients of  $|\psi'\rangle_{AB}$  are  $\lambda_{\pm} := [1 \pm \sqrt{1-x^2}]/2$ , and the concurrence of the input state is  $C(\Lambda_{|u_1|u_0|}^A(|\psi'\rangle_{AB})) = |\langle u_1|u_0\rangle|^{\frac{1-T}{2}}x$  from Ref. [24], where  $x := 2\sqrt{q_0q_1(1-|\langle u_1|u_0\rangle|^2)}$ . Hence,  $C^{\max}(P_s) = |\langle u_1|u_0\rangle|^{\frac{1-T}{2}}$  for  $P_s < 1 - \sqrt{1-x^2}$  and  $C^{\max}(P_s) = P_s^{-1}|\langle u_1|u_0\rangle|^{\frac{1-T}{2}}x[1 - 2(1-P_s)/(1+\sqrt{1-x^2})]^{1/2}$  for  $P_s \geq 1 - \sqrt{1-x^2}$ . Since  $C^{\max}(P_s)$  is a monotonically nondecreasing function of  $x$ , the choice of  $q_0 = q_1 = 1/2$  gives the maximum value of  $C^{\max}(P_s)$ , which is further bounded by an achievable concurrence  $C_{u^*}^{\text{opt}}(P_s)$  with

$$C_{u^*}^{\text{opt}}(P_s) := \frac{u^{\frac{1-T}{2}}\sqrt{(1-u)(2P_s+u-1)}}{P_s} \quad (1)$$

for

$$u^* := \frac{1}{2} \left[ (1-P_s)(2-T) + \sqrt{4P_s^2(1-T) + (1-P_s)^2T^2} \right] \quad (2)$$

satisfying  $1 - P_s \leq u^* \leq 1$ . Therefore, the performance  $(P_s, P_s\bar{E})$  of any protocol must be in the convex hull of  $\{(P_s, P_s\bar{E}) \mid 0 \leq P_s \leq 1, 0 \leq \bar{E} \leq E(C_{u^*}^{\text{opt}}(P_s))\}$ .

*Optimal protocol.*—We have shown that the achievable region of an arbitrary protocol is described by Eqs. (1) and (2). Here we present a specific protocol achieving the optimal bound  $C_{u^*}^{\text{opt}}(P_s)$  except for a trivial point  $P_s = 1$ . We allow Alice and Bob to use a realizable [7] interaction between an off-resonance laser pulse in a coherent state  $|\alpha\rangle_a$  and a matter qubit  $A$ , which is described by a unitary operation  $\hat{U}_{\theta}|j\rangle_A|\alpha\rangle_a = |j\rangle_A|\alpha e^{i(-1)^j\theta/2}\rangle_a$  for  $j = 0, 1$ .  $\theta$  depends on the strength of the interaction ( $\theta \sim 0.01$  [7]). Let us consider the following protocol [see Fig. 1 (a)]: (1) Alice makes a probe pulse in a coherent state  $|\alpha/\sqrt{T}\rangle_a$  ( $\alpha \geq 0$ ) interact with her qubit  $A$  in a state  $(\sum_{j=0,1} e^{-i(-1)^j\zeta_{\alpha}/\sqrt{T}}|j\rangle_A)/\sqrt{2}$  with  $\zeta_{\alpha} := (1/2)\alpha^2 \sin \theta$  by  $\hat{U}_{\theta}$ , and she applies a displacement operation  $\hat{D}_{-(\alpha/\sqrt{T})\cos(\theta/2)}$  to the pulse  $a$ ; (2) Alice sends the pulse to Bob through a lossy channel  $a \rightarrow b_1$  (with transmittance  $T$ ) together with the local oscillator (LO); (3) On receiving the pulse  $b_1$  and the LO, Bob generates a second probe pulse  $b_2$  in a coherent state  $|\beta\rangle_{b_2}$  with  $\beta \geq \alpha$  from the LO, and he makes the pulse  $b_2$  interact with his qubit  $B$  in state  $(\sum_{j=0,1} e^{-i(-1)^j\zeta_{\beta}}|j\rangle_B)/\sqrt{2}$  by  $\hat{U}_{\theta}$ ; (4) Bob applies a displacement operation  $\hat{D}_{-\beta\cos(\theta/2)}$  to the pulse  $b_2$ ; (5) Bob further applies a 50/50 beam splitter described by  $|\alpha_1\rangle_{b_1}|\alpha_2\rangle_{b_2} \rightarrow |(\alpha_1+\alpha_2)/\sqrt{2}\rangle_{b_3}|(\alpha_1-\alpha_2)/\sqrt{2}\rangle_{b_4}$  to the pulses in modes  $b_1$  and  $b_2$ ; (6) Bob applies a QND measurement to pulses  $b_3$  and  $b_4$  in order to execute a projective measurement  $\{\hat{Q}_s^{b_3b_4}, \hat{1}_{b_3b_4} - \hat{Q}_s^{b_3b_4}\}$  with  $\hat{Q}_s^{b_3b_4} := \hat{1}_{b_3b_4} - \sum_{n=0}^{\infty} |n\rangle\langle n|_{b_3} \otimes |n\rangle\langle n|_{b_4}$ ; (7) If Bob receives an outcome corresponding to the projection  $\hat{Q}_s^{b_3b_4}$ , Bob declares the success of the protocol.

In the virtual protocol for this scheme, since Bob's operations in steps (3)-(7) commute with the phase-flip channel  $\Lambda_{|u_1|u_0|}^A$ , the operations are assumed to be directly applied to the state  $|\psi'\rangle_{AB}$ . In this sense, the state after step (6) is described by  $|\chi\rangle_{ABb_3b_4} = [|00\rangle_{AB}|i\gamma_+\rangle_{b_3}|-i\gamma_-\rangle_{b_4} + |01\rangle_{AB}|-i\gamma_-\rangle_{b_3}|i\gamma_+\rangle_{b_4} + |10\rangle_{AB}|i\gamma_-\rangle_{b_3}|-i\gamma_+\rangle_{b_4} + |11\rangle_{AB}|-i\gamma_+\rangle_{b_3}|i\gamma_-\rangle_{b_4}]/2$  with  $\gamma_{\pm} := [(\beta \pm \alpha) \sin(\theta/2)]/\sqrt{2}$ . This state can be represented, in the respective phase spaces of modes  $b_3$  and  $b_4$ , by  $|\chi\rangle_{ABb_3b_4}$  in Fig. 1 (a). This figure suggests an intuitive reason why this protocol can generate entanglement between qubits  $AB$ : If there are more photons in mode  $b_3$  ( $b_4$ ) than in mode  $b_4$  ( $b_3$ ), the possibility that the state has lived in the subspace spanned by  $\{|00\rangle_{AB}, |11\rangle_{AB}\}$  ( $\{|01\rangle_{AB}, |10\rangle_{AB}\}$ ) is higher. A direct calculation shows  $\|_A \langle j | \hat{Q}_s^{b_3b_4} | \chi \rangle_{ABb_3b_4} \|^2 = [1 - e^{-\gamma_+^2 - \gamma_-^2} I_0(2\gamma_+ \gamma_-)]/2$  for  $j = 0, 1$  and  ${}_{ABb_3b_4} \langle \chi | (|1\rangle\langle 0|_A \otimes \hat{Q}_s^{b_3b_4}) | \chi \rangle_{ABb_3b_4} = [e^{-(\gamma_+ - \gamma_-)^2} - e^{-\gamma_+^2 - \gamma_-^2} I_0(2\gamma_+ \gamma_-)]/2$ , where  $I_0(x) := \sum_{n=0}^{\infty} (x/2)^{2n}/(n!)^2$  is a modified Bessel function. Thus, the success probability  $P_s$  is

$$P_s = 1 - e^{-(\beta^2 + \alpha^2) \sin^2(\theta/2)} I_0((\beta^2 - \alpha^2) \sin^2(\theta/2)). \quad (3)$$

In addition, since the final state is written  $\Lambda_{u_{\alpha}}^A(|\phi\rangle\langle\phi|_{ABb_3b_4})$  with  $|\phi\rangle_{ABb_3b_4} := \hat{Q}_s^{b_3b_4}|\chi\rangle_{ABb_3b_4}/\sqrt{P_s}$  and  $u_{\alpha} := e^{-2\alpha^2 \sin^2(\theta/2)}$ , it is concluded that the concurrence  $C$  between  $A$  and  $Bb_3b_4$  satisfies  $C(\Lambda_{u_{\alpha}}^A(|\phi\rangle\langle\phi|_{ABb_3b_4})) = C_{u_{\alpha}}^{\text{opt}}(P_s)$  from Ref. [24]. On the other hand, for any  $\alpha$  and  $P$  satisfying  $1 - u_{\alpha} \leq P < 1$ , there is a choice of  $\beta$  for making  $P_s = P$  hold. Hence, fixing  $P_s = P$ , we can choose  $\alpha$  such that  $u_{\alpha}$  is equivalent to  $u^*$  of Eq. (2). Thus, the present protocol attains the optimal performance  $C_{u^*}^{\text{opt}}(P_s)$ .

*Near-optimal protocol.*—We have shown that a protocol employing the QND measurement on incoming pulses can optimally generate entanglement between Alice's qubit  $A$  and Bob's entire system  $Bb_3b_4$  including pulses  $b_3b_4$ . However, in practice, it is difficult to achieve such a QND measurement, and the pulses  $b_3b_4$  are unsuitable for storing the entangled state for a long time. Therefore, it is important to find a protocol that does not need to use a QND measurement and produces entanglement between Alice and Bob's qubits  $AB$  instead of  $A$  and  $Bb_3b_4$ . One such protocol can be obtained by replacing steps (6) and (7) in the optimal protocol with the following steps [see Fig. 1 (a)]: (6') Bob counts the number of photons by using photon-number-resolving detectors in modes  $b_3$  and  $b_4$ , respectively; (7') If the outcomes  $m$  and  $n$  of the two detectors are different, Bob declares the success of the protocol. We consider this modified protocol below.

From the definition, the success probability  $P_s$  must be the same as Eq. (3). In the virtual protocol for this scheme, with probability  $P_{mn} := e^{-\gamma_+^2 - \gamma_-^2} (\gamma_+^{2m} \gamma_-^{2n} + \gamma_-^{2m} \gamma_+^{2n})/(2m!n!)$ , the protocol returns outcomes  $m$  and  $n$ , and provides a final state  $\Lambda_{u_{\alpha}}^A(|\phi_{mn}\rangle\langle\phi_{mn}|_{AB})$  for state



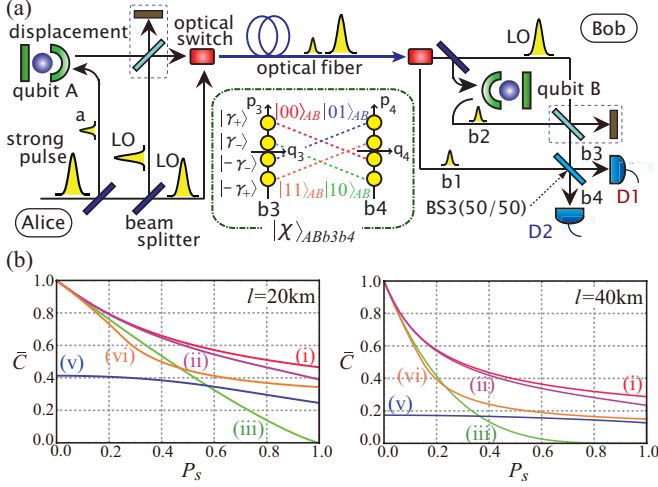


FIG. 1: (a) Schematic of near-optimal protocol. If we replace the photon detectors D1 and D2 with the QND measurement to perform the projection  $\hat{Q}_{s^{b_3b_4}}$ , we can reduce the protocol to the optimal one. (b) Performance of various protocols: The average concurrence  $\bar{C}$  as a function of the success probability  $P_s$  when  $T = e^{-l/l_0}$  with  $l_0 = 25\text{ km}$  ( $\sim 0.17\text{ dB/km}$  attenuation) and  $\theta = 0.01$ , for (i) the optimal protocol, (ii) the near-optimal protocol, (iii) a photon-detector-based two-probe protocol [10] that achieves a tight bound [13] for single-error-type entanglement generation, (iv) a photon-detector-based single-probe protocol [8, 9], and (v) a homodyne-detection-based single-probe protocol [7].

$|\phi_{mn}\rangle_{AB} := b_3\langle m|_b b_4\langle n|_b |\chi\rangle_{ABb_3b_4}/\sqrt{P_{mn}}$  with concurrence  $C(\Lambda_{u_\alpha}^A(|\phi_{mn}\rangle\langle\phi_{mn}|_{AB})) = u_\alpha^{\frac{1-T}{T}} e^{-\gamma_+^2 - \gamma_-^2} |\gamma_+^{2m} \gamma_-^{2n} - \gamma_-^{2m} \gamma_+^{2n}| / (2m!n!P_{mn})$  from Ref. [24]. Hence, for an entanglement monotone  $E$  with  $E(C)$ , the average of the entanglement monotones is determined by  $\bar{E} = [\sum_{m,n \geq 0} (1 - \delta_{mn}) P_{mn} E(C(\Lambda_{u_\alpha}^A(|\phi_{mn}\rangle\langle\phi_{mn}|_{AB})))] / P_s$ . Parameters  $\alpha$  and  $\beta$  (determining  $\gamma_\pm$ ) should be chosen to maximize  $\bar{E}$  with  $P_s$  fixed.

In Fig. 1 (b), we show the performance of several known protocols [7–10] as well as the optimal and near-optimal protocols in terms of the average concurrence  $\bar{C}$ . For comparison, we assume that all the devices used in the protocols are ideal. From the figures, we can confirm that the near-optimal protocol performs similarly to the optimal protocol and it outperforms the existing protocols [6–10]. Through the relation  $E = E(C)$  for qubits, one could also easily estimate the performance even in terms of the entanglement monotone  $E$ .

In conclusion, we have provided an optimal bound  $E(C_{u^*}^{\text{opt}}(P_s))$  defined by Eqs. (1) and (2) for arbitrary entanglement manipulation via coherent-state transmission. In addition, we have presented a simple optimal scheme and its practical version [Fig. 1 (a)] with almost optimal performance. This suggests that quantum optical devices in quantum communication can become as powerful as arbitrary operations. The setting of the prob-

lem respects a shared nature of known realistic schemes [5–11], but we believe that our solution to the problem will provide new insights into fundamental theories such as those in Refs. [4, 12, 17, 23] and into limits on other quantum communication protocols as in Refs. [25, 26].

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